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Chapter 13

Evidence for Cognitive Science Principles that Impact Learning in Mathematics

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INTRODUCTION

Students in the United States consistently underperform on state tests of mathematical proficiency (e.g., Kim, Schneider, Engec, & Siskind, 2006; Pennsylvania Department of Education, 2011) and in international comparisons on the Trends in International Mathematics and Science Study (TIMSS; e.g., Mullis, Martin, Foy, & Arora, 2012) and Programme for International Student Assessment (PISA; e.g., Fleischman, Hopstock, Pelczar, & Shelley, 2010). Among other issues, aspects of early mathematics instruction can interfere with later learning (McNeil et al., 2006; see also McNeil et al., Chapter 8; Van Hoof et al., Chapter 5) and students are not adequately prepared to tackle difficult gatekeepers, such as fractions (Booth & Newton, 2012) and algebra (Department of Education, 1997), that in turn prevent them from advancing in the fields of Science, Technology, Engineering, and Mathematics (STEM). To address these issues, educators and researchers have repeatedly called for mathematics instruction in the United States to be based on evidence (e.g., CCSSI, 2010; NCLB, 2002; NMAP, 2008).

Over the past few decades, the field of cognitive science has identified factors that have the potential to substantively improve students' learning. These principles have been detailed in several reviews (e.g., Dunlosky, Rawson, Marsh, Nathan, & Willingham, 2013; Koedinger, Booth, & Klahr, 2013; Pashler et al., 2007), and many involve a comparison of different approaches (such as spaced vs. massed practice), showing that one type of instructional technique

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is superior to another type of instructional technique (Koedinger et al., 2013). Many of these cognitive science principles are potentially useful for improving mathematics instruction, perhaps especially those concerning abstract and concrete representations, analogical comparison, feedback, error reflection, scaffolding, distributed practice, interleaved practice, and worked examples.

However, a vast divide persists between research and practice, as many of these principles are not consistently used in US classrooms. For example, mathematics textbooks in the UnitedStates frequently include decorative images alongside mathematically relevant images (Lehman, Schraw, McCrudden, & Hartley, 2007) and lack worked examples for students to study or explain (Mayer, Sims, & Tajika, 1995). Moreover, US teachers tend to shy away from talking about errors that could be useful for targeting students' conceptual misunderstandings (Lannin, Townsend, & Barker, 2006). Many of the recommended practices mimic those common in mathematics instruction in countries with traditionally higher mathematics achievement. For example, teachers in Hong Kong and Japan make more connections between abstract and concrete representations (Richland, Zur, & Holyoak, 2007), worked examples are more common in Japanese textbooks (Mayer et al., 1995), students are taught to use particularly effective concrete models in Singapore (Hoven & Garelick, 2007; see also Lee, Ng, & Bull, Chapter 9), and errors are considered to be critical opportunities for learning in Japanese classrooms (Stigler & Hiebert, 1999).

The fact that these practices are prominent in countries that consistently outperform the United States suggests these practices may also be useful in US classrooms. However, before the wholesale implementation of these practices, it is important to first decompose them into the underlying cognitive principles and then test them within the US educational system. This is important because experiences outside of the classroom, prior instruction, and parental expectations may also differ across countries and in ways that might influence the effectiveness of instructional techniques in different contexts.

In the present chapter, we review the evidence for several principles that are especially promising for improving mathematics instruction (Table 13.1). Using the classification scheme proposed by Koedinger et al. (2013), we begin with principles that focus on improving memory and fluency: scaffolding, distributed practice, and feedback. We then move to worked examples, interleaved practice, and abstract and concrete representations, which primarily promote induction and refinement. Finally, we review principles that are geared towards improving sense making and understanding: Error reflection and analogical comparison. In each section, we first describe the principle and the instructional implications. We then evaluate the evidence for the effectiveness of the principle for promoting mathematics learning, first from laboratory studies and then from classroom studies. Finally, we summarize what is and is not known about how these principles relate to mathematics learning, and identify gaps to be addressed in further translational research. The chapter concludes with an overall evaluation of the current state of evidence regarding how the use of these cognitive principles

TABLE 13.1 Cognitive Principles

Principle	Basic description
Scaffolding	Having instructional support gradually faded enables learners to solve problems fluently
Distributed practice	Spacing out practice leads to better memorization than practicing all at once
Feedback	Receiving feedback about responses improves achievement
Worked examples	Studying (or explaining) worked out examples plus solving problems is better for refining knowledge than problem solving alone
Interleaving	Practicing solving different types of problems in a mixed order is better for problem solving skill than practicing the same type of problems in a row.
Abstract and concrete representations	Linking between abstract and concrete representations yields increased learning and transfer
Error reflection	Thinking about errors improves problem representation and increases conceptual understanding
Analogical comparison	Comparing and contrasting multiple instances leads to better understanding than studying one instance

can influence mathematics instruction, both for simple and for more complex skills and concepts; we also suggest research priorities necessary to maximize the potential impact of cognitive science for mathematics instruction.

SCAFFOLDING PRINCIPLE

Scaffolding derives from Vygotsky's (1978) Zone of Proximal Development (ZPD), or the distance between the individual learner's current development and their potential development if guided by parents, teachers, or peers, within socio-cultural theory. With regard to learning, the theory suggests that adults play a role in children's problem solving activities while taking into account the child's level of understanding (Kupers, van Dijk, & van Geert, 2015). During the initial learning stages, adults offer large amounts of assistance and as the child becomes more competent, the adult removes assistance gradually. Under this assumption, learning begins as a social activity before it becomes an independent one. As an instructional strategy, scaffolding holds promise for promoting improved problem-solving performance (Kim & Hannafin, 2011). Broadly, scaffolding has been defined as any activity that involves modeling a specific behavior for a child or student, and then removing this modeling as the learner becomes more knowledgeable (Wood, Bruner, & Ross, 1976). More specifically, scaffolding involves three components: (1) scaffolding should be contingent on students' level of understanding (Pratt & Savoy-Levine, 1998), (2) there should be a gradual withdrawal of scaffolding (Calder, 2015), and (3) there should be a responsibility transfer from the teacher or adult to the student (van de Pol, Volman, & Beishuizen, 2010). Thus, not only should a teacher monitor the students' learning as they progress, but an initial assessment of the students' knowledge level is necessary to adapt scaffolding strategies for effective learning.

Evidence from Laboratory Studies

With respect to mathematics learning, laboratory studies demonstrate that considering students' prior knowledge about the content is a key component when determining how much or how little scaffolding is needed (Schwonke, Renkl, Salden, & Aleven, 2011; see also Thevenot, Chapter 3). Nevertheless, providing no scaffolding may be more effective than providing predetermined levels of scaffolding to all participants. In a metaanalysis comparing fixed scaffolds to those that were contingent on students' ZPD, Belland, Walker, Olsen, and Leary (2015) found that predetermined, fixed amounts of scaffolding delays the transfer of responsibility process likely because it did not account for the learners' current competence level and subsequently hampered the learning process.

Evidence from Classroom Studies

In mathematics classrooms, scaffolding has been shown to be necessary in early stages of learning particularly for students attempting to learn content just outside of their ZPD (Smit, van Eerde, & Bakker, 2013), but must be faded out successively in order to pass responsibility to the learner (Razzaq & Heffernan, 2006; Vorhölter, Kaiser, & Ferri, 2014). When scaffolds are not removed over time, combined with not taking into account the students' competence level, performance can suffer (Schwonke et al., 2011). As with laboratory work, the successful use of scaffolding in mathematics classrooms is based on individualized instruction through computer adaptive learning or one-on-one strategies (e.g., Salden, Aleven, Renkl, & Schwonke, 2009), rather than a one size fits all scaffolding approach.

Recommendations for Further Research

A lack of consensus on the definition of scaffolding has made assessing the effectiveness of this approach difficult. For example, proponents of Wood et al.'s (1976) perspective define scaffolding as support from a teacher or more competent peer to help students solve problems that are beyond their capacity, whereas van de Pol et al. (2010) require that scaffolding adjusts to the students' level of understanding, fading or successively removing scaffolds, and transferring responsibility from the teacher to the learner. Further, this definition has not yet been broadened to include scaffolds used in intelligent tutoring systems. In these computer-based contexts, the computer uses students' prior knowledge to determine the level of fading needed to adapt to the individual learner, making this activity less of a human-to-human interaction. As it stands, it is difficult to compare the effectiveness of this strategy across studies while also being aware that scaffolding does not look the same for every teacher-student interaction, and thus a general outline for what scaffolding is not may be an important part of reaching a consensus on what scaffolding includes (van de Pol et al., 2010).

Further, it seems clear that fading of support is necessary, but further work should investigate how and when it is best to fade for different mathematical content, not only by considering the learners' prior knowledge, but other cognitive characteristics (e.g., working memory, inhibitory control, and attention) that might improve or diminish learning if there are too few or too many scaffolds in place. Considering access to computer based learning, however, it may also be necessary to establish methods for providing faded support in traditional mathematics classrooms when personalized, computerized instruction is not in place, though this may not be feasible given the nature of scaffolds.

DISTRIBUTED PRACTICE EFFECT

Distributed practice is one of the most widely studied cognitive principles, whereby spreading out learning opportunities (spaced practice)—either within a single study session or across multiple sessions—leads to better long-term retention of studied material than does providing multiple learning opportunities one right after the other (massed practice) (Dunlosky et al., 2013). It is important to note that this principle does not refer to increasing the number of practice problems, but rather, the temporal distribution of those problems. As Rohrer (2009) clarifies,

> *For example, rather than work 10 problems on the same topic in one session, a student might divide the same 10 problems across two sessions separated by a week. Furthermore, spacing does not entail more recent practice, because in a well-designed spacing experiment, the interval between study and test, the test delay, is measured from the last practiced problem (Rohrer, 2009, p. 8).*

Research on distributed practice has yielded two types of effects, spacing, and lag effects, with implications for the distribution of both study time and of practice testing. A *spacing effect* refers to an advantage of spaced-out practice over massed, or blocked, practice (see Cepeda, Pashler, Vul, Wixted, & Rohrer, 2006 for a review). A *lag effect* refers to advantages of spacing with longer lags (i.e., longer time between practice sessions) versus spacing with shorter lags (i.e., shorter time between practice sessions) (Carpenter, Pashler, & Cepeda, 2009). If stimuli are being processed intentionally, spacing effects are larger than when processing is unintentional (Janiszewski, Noel, &

Sawyer, 2003); however, these effects can also be observed under conditions of incidental learning when deep processing occurs (Challis, 1993).

Evidence from Laboratory Studies

Within the domain of mathematics, laboratory studies are sparse, but available results indicate there are also benefits for spaced practice on tasks that require more than rote memorization. For instance, spaced practice is superior to massed practice for understanding abstract mathematical concepts, such as determining simple permutations (Rohrer & Taylor, 2006, 2007). Longer lags were also shown to be superior to shorter lags for learning basic arithmetic facts (Rickard, Lau, & Pashler, 2008).

Evidence from Classroom Studies

Within mathematics classrooms, benefits of spaced practice have been observed for fluent retrieval of both addition (Schutte et al., 2015) and multiplication facts (Rea & Modigliani, 1985). Some classroom studies have focused on more complex tasks, but have also often conflated testing of multiple cognitive principles. Thus, it is hard to isolate the benefit of spaced practice in mathematics for these more complex tasks. For example, Gay (1973) claimed to find greater benefit for spaced practice in algebra classrooms, but spacing was confounded with test delay in the study (Rohrer, 2009). A classroom study examining conceptual understanding in statistics, however, did confirm that longer lags between practice sessions were superior to shorter lags (Budé, Imbos, van de Wiel, & Berger, 2011)

Recommendations for Further Research

As noted by a number of researchers, there is a lack of studies exploring the effects of spacing of instruction or practice on mathematics learning. Spaced practice is not widely evident in mathematics instruction (Bahrick & Hall, 2005; Dempster, 1988; Mayfield & Chase, 2002; Rohrer, 2009; Willingham, 2002), and mathematics textbooks often promote blocked practice rather than spaced practice (Rohrer, 2009). Further study is needed to evaluate the effect of distributed practice on a variety of mathematics skills, especially those beyond rote fact memorization.

Another clear gap in the research is whether the effects of distributed practice vary with individual differences in learner characteristics. Several potential characteristics of interest have been suggested, including prior knowledge, and motivation (Dunlosky et al., 2013), and cognitive capacities (Delaney, Verkoeijen, & Spirgel, 2010). However, there have been no direct tests of whether the benefits of spacing and lag effects in mathematics learning are moderated by these individual differences variables.

Delaney et al. (2010) suggest that researchers must examine whether other classroom activities may be effectively spaced to achieve the same goals. For example, classroom discussions may similarly serve to remind students of the content, and distributing discussion of those topics over time may be even more effective than distributing practice on those ideas over time (Delaney et al., 2010).

FEEDBACK PRINCIPLE

Generally, feedback is considered to be any form of information whose purpose is to enlighten a recipient (Mory, 2004). The study of feedback—in the form of reinforcement—dates back to Skinner's assertion that it reinforces behavior (Kulhavy, 1977), but feedback itself is considered to be "instructionally powerful" (Cohen, 1985, p. 33) for many contexts, not just behavior. Within a learning context, feedback is typically given in response to an instructional question or task and is intended to improve the learners' problem solving accuracy or conceptual understanding of the topic (Carter, 1984; Kulhavy, 1977). The effectiveness of feedback has been widely studied (see Shute, 2008 for a review), though not all students interpret teacher feedback in the same way (Williams, 1997). Recent research has focused specifically on the nature and timing of the feedback given, and how those factors impact its effectiveness.

Evidence from Laboratory Studies

For mathematics learning, one laboratory study showed that feedback helps to promote the formation of strategies used to solve arithmetic problems (Alibali, 1999). Further, process-oriented feedback (i.e., individualized comments about a student's confidence, strengths, and weaknesses) is more helpful than grade-oriented feedback (i.e., assigning a score, based on student performance compared to a reference group) for improving both mathematics achievement and interest in 9th grade students (Harks, Rakoczy, Hattie, Besser, & Klieme, 2014).

Students with low prior ability or knowledge may benefit more strongly from either outcome or strategy based feedback than their higher ability or more knowledgeable peers (Fyfe, Rittle-Johnson, & DeCaro, 2012; Krause, Stark, & Mandl, 2009; Luwel, Foustana, Papadatos, & Verschaffel, 2011). A student's working memory capacity may also moderate the type of feedback that is most beneficial; for instance, outcome-based feedback may be more beneficial for students with a lower working memory capacity, as more complicated feedback may tax their already limited working memory resources, while strategy and outcome-based feedback may be equally helpful for students with a higher working memory capacity (Fyfe, DeCaro, & Rittle-Johnson, 2015).

Evidence from Classroom Studies

Classroom studies providing individualized feedback have confirmed that immediate feedback is better than delayed feedback (Kehrer, Kelly, & Heffernan, 2013; Singh et al., 2011) and that it can increase the development of expertise (Ellis, Klahr, & Siegler, 1993). Further, video feedback (e.g., an individual providing verbal explanation feedback while referring to a whiteboard illustration) may be even better than traditional textual feedback as it was found to slow down the pace of the learner, suggestively allowing more time for the student to internalize the concept (Ostrow & Heffernan, 2014).

Recommendations for Further Research

In classroom settings, information about the effectiveness of method of delivery and frequency of immediate, detailed, process-based feedback is limited. It remains to be determined how to best individualize feedback in classroom settings, and whether the types of feedback provided by computer assisted learning can be provided by teachers in classroom settings. In addition, more work is needed to test the effectiveness of the kinds of feedback that are typically given in the context of traditional classrooms that are not technology-based. Finally, more research is needed to assess the effects of negative feedback in terms of potential improvements in learning without undermining motivation.

WORKED EXAMPLE PRINCIPLE

The worked example principle suggests that having learners study examples of worked-out solutions to problems is more effective for learning than having them solve all of the problems themselves (Sweller & Cooper, 1985). Further learning benefits are found when the learners are asked to explain the examples (Renkl, Stark, Gruber, & Mandl, 1998; Hausmann & van Lehn, 2007). Several mechanisms have been invoked to explain this principle. First, studying worked examples is thought to reduce learners' cognitive load by reducing the attentional and working memory demands needed to remember all of the problem solving steps (see also Lee, Ng, & Bull, Chapter 9). Instead, they can focus their limited working memory capacity on understanding the reasoning behind the procedural steps taken in the example. Second, prompting learners to *explain* the steps in the example provides the additional benefits inherent to self-explanation, including making knowledge explicit, connecting new information to prior knowledge, and generating inferences to fill knowledge gaps (Roy & Chi, 2005). Studying and explaining worked examples may be most beneficial for novice than expert learners (Booth et al., 2015b; Kalyuga, Chandler, Tuovinen, & Sweller, 2001).

Evidence from Laboratory Studies

The worked example principle has been successfully applied in a number of laboratory studies within the domain of mathematics. These studies demonstrate that studying worked examples increases problem-solving performance or transfer for undergraduate students learning probability (, 1996, 1998) and business math (Hsiao, Hung, Lan, & Jeng, 2013), as well as high school students learning algebra (Cooper & Sweller, 1987; Sweller & Cooper, 1985), and geometry (Tarmizi & Sweller, 1988). However, these benefits may only be attained when the worked examples do not require learners to split their attention between two separate sources of information (Tarmizi & Sweller, 1988; see also Rittle-Johnson, Star, & Durkin, Chapter 12). Learners may also experience a decrease in the amount of time it takes them to learn mathematical content (Zhu & Simon, 1987) or to solve problems on their own (Sweller & Cooper, 1985).

Evidence from Classroom Studies

Most studies of the worked examples effect that have taken place in the context of real-world courses have relied on computer-assisted instruction. Studies conducted in computerized mathematics classrooms have revealed benefits including a decrease in the amount of time it takes high school students to learn in geometry (Schwonke et al., 2007), an increase in procedural performance for 5th graders learning about equivalent fractions (Lee & Chen, 2015), and increased performance and improved attitudes for middle school students learning basic mathematical concepts via worked example-based podcasts (Kay & Edwards, 2012). Benefits for conceptual understanding have also been found for middle and high-school algebra students explaining worked examples in computerized settings (Booth, Lange, Koedinger, & Newton, 2013; Reed, Corbett, Hoffman, Wagner, & MacLaren, 2013).

A growing number of worked examples studies have been conducted in more traditional classrooms. A study conducted in a high school algebra classroom found that students who studied worked examples made fewer problem solving errors, took less time to learn, and required less assistance from their teacher (Carroll, 1994). Longer-term classroom studies have shown a time-on-task benefit for Dutch 4th graders who studied worked examples while learning about subtraction (van Loon-Hillen, van Gog, & Brand-Gruwel, 2012) and benefits for conceptual and procedural performance as well as performance on standardized test items for middle and high school students who explained correct and incorrect examples while learning algebra (Booth et al., 2015a); however, benefits may be observed only for students with some prior knowledge who are not yet experts (Booth et al., 2015b). Finally, Retnowati, Ayres, and Sweller (2010) found that 7th graders improved their understanding of how to solve geometry problems after studying worked examples either individually or in group work; however, metacognitive training including comprehension, connection, and reflection questions while solving problems may be even more effective than studying worked examples for middle school students working in groups (Mevarech & Kramarski, 2003).

Recommendations for Further Research

Recent studies have tested the effectiveness of worked examples in computerized and real-world classrooms. While these studies typically demonstrate benefits of studying and explaining worked examples, there are published exceptions where the intervention does not work, or does not work for all students (e.g., Booth et al., 2015b). Research is needed to investigate the conditions under which worked examples are beneficial in educational settings. For instance, does the benefit of worked examples vary based on the particular mathematical content to be learned? Is feedback necessary when students self-explain examples? To what degree does the formatting of the example matter? And what level of detailed translational work is necessary to ensure that worked examples will be beneficial in real-world classrooms?

Also, while it has been established that prior knowledge impacts learning from worked examples, all studies to date have examined general achievement or prior knowledge of the particular topic the students are learning about. However, might the type of prior knowledge that is important for learning vary with different mathematical topics? For example, are there particular types of prerequisite knowledge or skills that might make students better prepared to learn a certain topic? If so, do students who lack that particular knowledge or skill benefit more from studying and explaining worked examples? Further investigation is needed to address these questions (see Rittle-Johnson, Star, & Durkin, Chapter 12).

INTERLEAVING PRINCIPLE

The interleaving principle suggests that when practice problems are alternated, with a problem on one concept followed by a problem on another concept, students learn better than if problems are blocked, or grouped by concept (Rohrer, 2012). Interleaving problem types is thought to be effective because it helps students to differentiate between problems of varying types and identify problem features that suggest different strategies for solution (Rohrer, 2012). It also may help students to build strong associations between problem types and appropriate solution strategies (Rohrer, Dedrick, & Burgess, 2014); in contrast, blocked practice allows students to solve problems by carrying out the same strategy on a group of problems without building such associations.

Evidence from Laboratory Studies

Mayfield and Chase (2002) are given credit for being the first to test the interleaving effect for mathematics learning. In their study, college students were given remediation on algebraic concepts, and those who received their problems in an interleaved format outscored the blocked group on application and problem-solving test items. However, the study confounded interleaving with spaced practice, as the interleaving group also had more space between practice problems of the same type. Subsequent work has investigated the effectiveness of interleaving independent of spacing. In two laboratory studies, college students were given interleaved practice for solving volume problems with different types of obscure solids and these students outscored students given blocked practice on the posttest (Rohrer & Taylor, 2007; LeBlanc & Simon, 2008). Taylor and Rohrer (2010) found that fourth grade students did not benefit immediately from interleaved problems on prisms, but outperformed peers who had experienced blocked practice at the follow-up test; blocked students often made errors by using strategies that were appropriate for a different problem type than they had been taught. Finally, using the SimStudent, an intelligent agent that is able to learn from demonstration and problem-solving experiences, Li, Cohen, and Koedinger (2012) found that interleaving allowed more opportunities for the simulated student to detect its errors and refine its knowledge on several different mathematical content areas (fractions, addition, and equation solving).

Evidence from Classroom Studies

In the last few years, several studies have tested the effectiveness of interleaved practice in real world mathematics classrooms. Ostrow, Heffernan, Heffernan, and Peterson (2015) recently conducted a study to directly replicate Rohrer and Taylor's (2007) results, but in a real-world instructional context. Using the ASSISTments system for homework assignments, Ostrow et al. (2015) found a positive interleaving effect for seventh grade students reviewing mathematics concepts, such as complementary and supplementary angles, surface area of pyramids, and probability of compound events. Rohrer, Dedrick, and Stershic (2015) also found that interleaving practice was more effective than blocked practice on immediate and delayed posttests, even when the blocked practice group was given an additional review period to control the delay between practice and tests.

Finally, other studies have replicated and extended laboratory findings in classroom settings, examining different features of problems that could be interleaved. For instance, one classroom study compared interleaving of task types with interleaving of the types of representations used to solve fraction problems; interleaving of task types was found to be more important than interleaving representations for 5th and 6th grade students learning about fractions (Rau, Aleven, & Rummel, 2013). Though most interleaving is conducted with problems that have some commonalities, Rohrer et al. (2014) found that interleaving problems was still effective even when problem types were completely dissimilar.

Recommendations for Further Research

Both laboratory and classroom studies have shown that interleaving problem types is beneficial for students of different ages and for learning about a wide variety of mathematics concepts. The benefit may emerge immediately or at a delay, and may be robust regardless of the similarities and differences of the problems being interleaved (Rohrer et al., 2014); further studies must be conducted to verify this robustness across varying mathematical topics.

In general, though results are positive, application of the interleaving effect to mathematics is relatively new and is being studied by only a few research groups. Perhaps because of this, classroom research is still relatively limited, and mathematics textbooks still support blocked rather than interleaved practice (Rohrer, 2012). Further research will thus clearly be required to make appreciable differences in mathematical practice. Future work must also investigate how blocked and interleaved practice might be effectively combined to enhance student learning.

ABSTRACT AND CONCRETE REPRESENTATIONS PRINCIPLES

Do students benefit more from a concrete representation of a concept, such as a physical object or contextualized story, or an abstract representation of that concept, such as a symbol or an equation? Though some early studies pitted the two types of representations against one another, and reported benefits of one type over the other, a larger body of work finds that each type of representation has unique advantages and students learn most effectively by making connections between those representations (Pashler et al., 2007). Concrete representations are thought to ground new information with prior knowledge, and abstract representations help learners to transfer this new knowledge flexibly to other situations (see Belenky & Schalk, 2014, for a review). Beyond just linking the two types of representations, *concreteness–fading* (i.e., progressively presenting students with concrete representations that become increasingly abstract) may support efficient learning (Goldstone & Son, 2005).

Evidence from Laboratory Studies

Within the domain of mathematics, laboratory studies across a wide variety of age groups suggest that concrete representations support initial understanding but may not lead to transfer (e.g., Kaminski, Sloutsky, & Heckler, 2005). In contrast, abstract representations can support transfer, but may not always promote conceptual understanding (De Bock, Deprez, Van Dooran, Roelens, & Verschaffel, 2011). Integrating concrete and abstract representations may provide synergistic benefits. For example, Uttal et al. (2013) found that for 7-year-old learning 2-digit arithmetic, highlighting the connections between manipulatives and symbolic, written problem representations help children both gain a sound conceptual understanding and transferable knowledge. Gestures can also be used to highlight similarities and differences between mathematics problems of different types (Richland & McDonough, 2010). In this study, students learned more about the relationship between permutation (e.g., the number of possible outcomes for gold and silver medals in a race with four runners) and combination problems (e.g., the number of ways four students could win two show tickets) when the instructor left both types of problem on the board and gestured back and forth between the two problems when explaining the similarities (the same number of participants, two winners) and differences (order of winning matters in the race, but not in winning tickets to a show).

In general, starting with concrete or grounded representations and fading to more abstract or formal representations seem to lead to the best learning and transfer (Braithwaite & Goldstone, 2013; McNeil & Fyfe, 2012). However, concrete representations with features irrelevant to the concept or task can be harmful; Uttal, Scudder, and DeLoache (1997) found that children have a hard time understanding that toy-like mathematics manipulatives could be both physical objects and representations of an abstract concept, and thus tended not to transfer learning to formal representations. Similarly, undergraduates working with abstract representations (i.e., generic shapes) learned more than students given concrete representations that included seductive details (i.e., representations of measuring cups of liquid when the learning objective was mathematical grouping) (Kaminski, Sloutsky, & Heckler, 2008). Even discrete objects that are individually relevant to the task may fail to improve learning if their *quantity* is irrelevant to the task (Kaminski & Sloutsky, 2013). Abstract representations may be more beneficial than concrete representations when problems are particularly complex; as shown for both elementary school students (McNeil & Uttal, 2009) and undergraduates (Koedinger, Alibali, & Nathan, 2008). Similarly, learners with low prior knowledge may benefit more from concrete, well-grounded materials (Braithwaite & Goldstone, 2015; Petersen & McNeil, 2013), while both older and higher prior knowledge students may be able to benefit from more abstract materials (Booth & Koedinger, 2012).

Evidence from Classroom Studies

Classroom studies in mathematics are more abundant for this principle than those discussed thus far, and are consistent with the idea that making connections between abstract and concrete representations is helpful for a wide range of age groups (e.g., Ainsworth, Bibby, & Wood, 2002; Bottge, 1999; Stephan & Akyuz, 2012). Starting with more concrete representations and fading towards more abstract ones was also found to be beneficial for learning in elementary school mathematics classrooms (Fyfe, McNeil, & Borjas, 2015). Overall, the effectiveness of concrete and abstract representations may vary with learner characteristics including age and prior knowledge, which are often confounded. Siler and Willows (2014) found that 8th grade students benefited from abstract representations, yet 6th grade students needed to see concrete representations in order to be successful with far transfer on the same problems. As they found that concrete representations were also beneficial for students with poor reasoning skills, it's possible that some age-related findings may stem from knowledge or skill differences.

Recommendations for Further Research

Future research on this principle should include more comprehensive examination of which type, combination, and ordering of representations are most beneficial for different learning goals and different learners. We need more precise and consistent ways of evaluating representations, learning goals, and learner characteristics. Apparently discrepant results regarding the effectiveness of interventions based on this principle could in fact be due to the operational definitions; for example, of *concrete* and *abstract* used in the studies, or the fact that prior knowledge affects whether a given representation is concrete or abstract for an individual learner. Prior knowledge and task familiarity may also underpin age- and grade-related trends that show differential benefits for concrete or abstract representations or from concreteness fading. Future research that operationalizes key factors related to the learner, the representations, the tasks, and learning goals, will yield more generalizable recommendations that inform practice and will enable practitioners to select the best representations or sequence of representations for their students and classroom objectives.

ERROR REFLECTION PRINCIPLE

The error reflection principle grew out of research on cognitive dissonance theory, originally proposed by Festinger (1957) to explain a series of findings from experiments on self-justification. The original theory proposed that a natural human motivation is to seek consistency amongst cognitions (i.e., thoughts, beliefs, or attitudes). *Cognitive dissonance* is thus a state of tension or discomfort that arises whenever one holds two cognitions that are inconsistent with one another (Festinger, 1957). While cognitive dissonance theory is often used to explain findings in social psychology on individuals' attitude change surrounding social topics (e.g., politics, stereotypes), one can see the great range of applicability of the theory to the study of cognitive change within learning environments. If experiencing cognitive dissonance can potentially motivate an individual to alter their cognitions to make them more consistent, then it follows that purposefully creating cognitive disequilibrium can effectively produce changes in students' thinking (Graesser, 2009). This idea can also be traced back to Piaget who argued that confronting discrepant ideas when in a state of disequilibrium is not only helpful but *essential* for knowledge growth (Piaget, 1980). Overoye and Storm (2015) review literature suggesting that learning tasks are particularly beneficial when they induce uncertainty in students and provoke them to attempt to resolve this uncertainty, such as presenting information contradictory to the learners' current knowledge.

There are a number of methods that can induce cognitive dissonance, including creating confusion (D'Mello & Graesser, 2014) and presenting students with discrepant events (Gorsky & Finegold, 1994). However, we focus here on a facet of cognitive dissonance that is particularly relevant to mathematics: reflection on errors—either one's own errors (Cherepinsky, 2011) or those of other learners as presented in the form of incorrect worked examples (Große & Renkl, 2007). Ohlsson's (1996) theory suggests that learning from errors can be particularly effective if learners are prompted to identify what features of the problem make the specific step taken incorrect, which can subsequently be used to correct faulty knowledge and fine-tune problem-solving. Studying errors is also thought to be beneficial because it provides exposure to multiple perspectives rather than just one's own perspective (Siegler & Chen, 2008).

Evidence from Laboratory Studies

One laboratory study within the domain of mathematics demonstrated that explaining *a combination* of correct and incorrect examples was superior for elementary students' learning about mathematical equality, compared to explaining correct examples alone (Siegler, 2002). However, Große and Renkl (2007) found that for undergraduate students learning about probability, those with lower prior knowledge were less able to benefit from incorrect examples, unless the segment of the solution that contained the error was highlighted.

Evidence from Classroom Studies

A growing number of classroom studies have demonstrated the effectiveness of reflecting on errors for mathematics learning. For instance, after being asked to correct errors in problems marked incorrect on their exams, undergraduate calculus students felt they gained a better understanding of the material (Cherepinsky, 2011). Having middle school algebra students explain incorrect examples led to improvement in their ability to encode algebraic equations (Booth et al., 2013); explaining a combination of correct and incorrect examples has also been shown to improve algebra students' conceptual and procedural knowledge (Booth et al., 2015a). A series of studies on students learning about decimal magnitudes have also demonstrated benefits for comparing correct and incorrect worked examples (Durkin & Rittle-Johnson, 2012), using incorrect worked examples within the context of a computerized tutoring intervention (Adams et al., 2014), and finding, explaining, and fixing errors within erroneous examples (McLaren, Adams, & Mayer, 2015); however, in each of these studies, benefits only emerged after a delay, not immediately after the activities.

Though many studies show the benefits of studying errors, there is less consensus on whether different groups of students experience differential benefits. For example, Heemsoth and Heinze (2014) found that students with high prior knowledge benefit more from identifying, explaining, and correcting errors than their less knowledgeable peers. In contrast, other studies have found that students with low prior knowledge benefit the most from studying and explaining errors (Barbieri & Booth, 2016; Booth et al., 2015b). Still others demonstrate that incorrect examples were equally effective for all students regardless of prior knowledge (Adams et al., 2014; Durkin & Rittle-Johnson, 2012). Thus, the degree of benefit from reflecting on errors may vary based on the particular activities required of the student, individual differences in prior knowledge, or perhaps even the particular mathematics domain. Though not previously explored, individual differences in cognitive competencies, such as working memory may also influence the effectiveness of reflecting on errors.

Recommendations for Further Research

The next step is to begin investigating and comparing particular features within error reflection activities. For example, do learners need to find, explain, and correct errors to achieve maximum benefit or would studying them be sufficient? Would comparing incorrect examples to correct ones be more effective than just studying and explaining incorrect examples alone? And does each of these types of error reflection activities have the potential to influence different types of knowledge or skills and at different time scales? Research must move from establishing the effectiveness of learning from errors in general to clarify how the two proposed mechanisms for benefits of error reflection map to particular task choices and resulting benefits for different groups of students.

ANALOGICAL COMPARISON PRINCIPLE

Analogical reasoning refers to the human ability to draw connections and notice contrasts between the relationships of two or more representations (see also DeWolf et al., Chapter 7). In the process of comparison, learners have to map between two relationships of representations based on alignments (or misalignments), leading to generalizable knowledge (Gentner, 1983; Gick & Holyoak, 1980). For an analogy to work, mental representations must share a common relation, but need not appear similar. For instance, two equations may be presented in a similar form (e.g. “3 + 4 = ?” and “5 + 3 = ?”), or may be presented in a dissimilar form (e.g., “3 + 4= ?” and “Johnny had five apples and Mary gave him three more”), however, the arithmetical relation between the two equations needs to correspond. The effectiveness of learning from analogies is known to vary with individual differences in executive function (EF); in particular, working memory and inhibitory control deficits make learning from comparisons difficult (Cho, Holyoak, & Cannon, 2007). Perhaps at least in part because prior knowledge offloads EF demands (Grossnickle, Dumas, Alexander, & Baggetta, 2016), prior knowledge also impacts students' ability to learn from comparisons (Gentner & Rattermann, 1991).

Evidence from Laboratory Studies

Within the domain of mathematics, laboratory research has confirmed that comparing multiple examples facilitates schema formation (see also Rittle-Johnson, Star, & Durkin, Chapter 12). This has been found for a variety of mathematical content areas including algebra (Novick & Holyoak, 1991), probability (Ross, 1989), number sense (Thompson & Opfer, 2010), ratio (Novick & Holyoak, 1991), statistics (Cummins, 1992), and equivalence (Hattikudur & Alibali, 2010). Further, providing instructional supports, such as visual alignment or linking gestures have been shown to facilitate learning from comparisons (Richland & McDonough, 2010).

Evidence from Classroom Studies

Classroom studies testing the analogical comparison principle have frequently shown benefits for conceptual understanding in a variety of mathematical content areas, including algebra (Rittle-Johnson & Star, 2009), computational estimation (Star & Rittle-Johnson, 2009), geometry (Guo & Pang, 2011), proportional reasoning (Begolli & Richland, 2016), numerical density (Vamvakoussi & Vosniadou, 2012), and division of natural and rational numbers (Richland & Hansen, 2013). For supporting conceptual understanding, Rittle-Johnson and Star (2009) found that comparing multiple solutions to a single problem was more beneficial than comparing multiple problem types using a single solution strategy, or comparing equivalent problem types with the same solution (Rittle-Johnson, Star, & Durkin, Chapter 12). Benefits of analogical comparison have also been found for procedural fluency and flexibility (Begolli & Richland, 2016; Star & Rittle-Johnson, 2009). Several types of instructional supports have been shown to facilitate learning from comparing multiple representations, including visually aligning the to-be-compared examples (Star & Rittle-Johnson, 2009), using linking gestures to help students notice critical features (Alibali & Nathan, 2007), and prompting exploration (Schwartz & Bransford, 1998) or providing conceptual instruction (Fyfe, DeCaro, & Rittle-Johnson, 2014) prior to studying contrasting cases.

Individual differences in effectiveness have also been found, as students with greater prior knowledge become more likely to notice critical features of the to-be-compared items (Rittle-Johnson, Star, & Durkin, 2009; Star & Rittle-Johnson, 2009). However, novices have also been found to benefit from comparison (Guo & Pang, 2011; Rittle-Johnson, Star, & Durkin, 2012). In particular, while students with high prior knowledge may benefit more from comparing solution methods, students with low prior knowledge may benefit more from sequential presentation or from comparing different problem types (Rittle-Johnson et al., 2009).

Recommendations for Further Research

Orchestrating instruction around comparisons remains a challenge for mathematics teachers (Ball, 1993; Stein, Engle, Smith, & Hughes, 2008). Teachers in the United States regularly use comparisons in their lessons, but they do not always provide necessary instructional supports (e.g., visually aligning source and target, using gestures) compared to teachers in countries with higher achieving students, such as Hong Kong and Japan (Richland et al., 2007). US teachers often try to control the analogies and only engage students in procedural aspects of the analogy, not requiring them to attend to the structural alignments between representations in order to participate in instructional discussions (Richland, Holyoak, & Stigler, 2004). This may lead to failed opportunities for students to learn the deep aspects of concepts and transferable knowledge that are typically found in laboratory studies. Thus, more translational research is needed to find ways to help teachers utilize analogical comparisons in their classrooms by examining year-long curriculum implementations of both interactional teaching strategies and instructional materials. A recent attempt suggests that a single 1-week professional development course seems to be insufficient for consistent uptake of instructional materials throughout a full curriculum, leading to small improvements in procedural knowledge (Star et al., 2015). Yet, these are important first steps towards testing and making instructional materials widely available for scalable use. Future scale-up studies need to systematically examine teacher-led discussions around instructional comparisons.

CONCLUSIONS AND FUTURE DIRECTIONS

Each of the principles reviewed in this chapter has a clear relevance to mathematics instruction and the potential to improve student learning. However, the principles vary widely in terms of the quantity and quality of the empirical support for their effectiveness for mathematics learning. For example, distributed practice and interleaving effects have been established in other content areas but have seldom been tested in mathematics contexts. These principles clearly need more attention, with studies conducted either in laboratory or classroom settings, to determine if and when they are effective for students' learning of different types of mathematical content. Other principles have been well-tested in laboratory studies, but need more investigations to determine when and how they are effective in real-world educational settings. For example, there is a growing body of evidence supporting error reflection, but less work detailing the benefits of different ways that errors can and should be used in the classroom. For certain principles, such as scaffolding and feedback, it is vital to focus on implementation and testing in more traditional mathematics classes where teachers cannot rely on aid of individualized technology. Finally, the effectiveness of many of the principles has been relatively well established in laboratory and classroom studies, including abstract/concrete representations, analogical comparison, and worked examples. Research on these principles needs to shift focus to understanding individual differences in the principle's effectiveness for learning different types of mathematics content, as well as nuances of implementation in real-world classrooms.

The principles also vary in terms of their effectiveness for simpler versus more complex mathematical content. For instance, distributed practice has been repeatedly shown to be effective for improving simple mathematics skills, such as memorizing arithmetic facts. However, research has not yet established whether and how distributed practice can be implemented to effectively increase learning of more complex mathematical concepts and skills. The distinction and connections between abstract and concrete representations may also vary depending on whether simple or complex content is to be learned. In particular, learning of complex content may require the inclusion of abstract representations, where simple content can be learned from concrete representations as long as they do not contain seductive details, such as decorative graphics. Still other principles have been shown to be effective for both simple and complex mathematical skills. For example, worked examples have been shown to be effective for simple content, such as basic mathematics skills as well as more complex content like probability and algebra. Similarly, scaffolding has been shown to enhance early stages of learning of both simple and complex content; feedback, interleaving, analogical comparison, and error reflection also have benefits for learning regardless of the type of content.

Finally, these principles also vary in terms of how difficult it is to implement them in real classrooms, with and without technology. For instance, concrete and abstract representations are already often incorporated into mathematics textbooks, though further efforts could be made in selecting appropriate graphics and determining how those components should be configured to minimize demands on cognitive load. Error reflection, analogical comparison, and interleaved practice can be incorporated into lessons via either interactive activities or in static, paper-based materials; though worked examples have most frequently been used in computer-based classrooms, methods for implementing them with paper and pencil in traditional classrooms have also been successful. However, distributed practice, scaffolding, and feedback may be particularly difficult to implement and test systematically in the context of noncomputerized classrooms. What do these principles even look like in traditional classrooms, when they cannot be provided with individualized computer programs? Can they be incorporated into written materials, or must they be actively implemented by teachers?

Recent work in the National Center for Cognition and Mathematics Instruction undertook the translation of several of these principles to redesign a popular middle school mathematics curriculum, Connected Mathematics 2 (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). In this project, worked examples were carefully designed to enhance the different kinds of practice problems inherent in the reform curriculum, and efforts were made to reduce graphics with seductive details and increase the mathematical relevance of all concrete representations included in the text. However, the other two included principles—distributed practice and feedback in terms of formative assessment—were trickier to implement and required supplemental guidebooks for teachers. Effective practice distribution in this case, without the aid of technology, may require a thorough understanding of what skills are being practiced in each problem type, how frequently they are revisited in different formats, and how to help students make connections between previously learned skills and ones that are upcoming (Davenport, Lepori, Hauk, Viviani, & Schneider, 2012). In this type of translational work, it is critical to understand what the enacted principle would look like in a classroom, which is not necessarily as straightforward as it seems from laboratory studies.

General Recommendations

Translational work is essential to ensure that relevant findings from basic science reach their potential to improve our education system (Koedinger et al., 2013). Publishing findings of cognitive principles and speculating about their implications may move research forward, but it rarely yields appreciable change in real-world classrooms. Few practitioners have access to academic journals, thus they are not frequently exposed to new research findings. Further, even knowing that something works in laboratory studies does not mean it will necessarily work in the classroom. Examples can be found in and outside of mathematics of laboratory-tested techniques failing to translate in classrooms (Davenport, Klahr, & Koedinger, 2007; Dynarski et al., 2007) or helping only a portion of the students in the class (Booth et al., 2015b). As classrooms are much less controlled than university laboratories, the work of translating laboratory findings to be applicable in the classroom requires intimate knowledge of both the principles and the education system, a willingness to develop lessons and materials iteratively, with many opportunities for revision as they are field-tested, and creative experimental design. Translational research may be best facilitated in collaboration with teachers and administrators from school districts to ensure that the development of materials and practices based on these principles yields classroom-ready products.

However, the best instruction is not one size fits all. Individual differences (e.g., prior knowledge, working memory capacity) may interact with the utility of different strategies. Differentiated instruction is heavily promoted in today's education system, and we need to be able to make good recommendations to practitioners about what techniques to use, when, and for whom. Studies on individual differences in effectiveness of instruction have mainly explored prior knowledge and cognitive factors; future research should further distinguish these factors and also investigate motivational characteristics that may impact whether a student benefits from a particular technique. We need to consider individual differences in demographic characteristics as well, especially given that factors like gender and race may influence the way in which instructional materials are viewed, interpreted, and accessed (Moreno & Flowerday, 2006). Continued investigation of individual differences is necessary in order to maximize the benefits of research findings for real-world educational settings.

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REFERENCES

Adams, D. M., McLaren, B. M., Durkin, K., Mayer, R. E., Rittle-Johnson, B., Isotani, S., & van Velsen, M. (2014). Using erroneous examples to improve mathematics learning with a web-based tutoring system. *Computers in Human Behavior*, *36*, 401–411.

Ainsworth, S., Bibby, P., & Wood, D. (2002). Examining the effects of different multiple representational systems in learning primary mathematics. *The Journal of the Learning Sciences*, *11*(1), 25–61.

Alibali, M. W. (1999). How children change their minds: strategy change can be gradual or abrupt. *Developmental Psychology*, *35*(1), 127–145.

Alibali, M. W., & Nathan, M. J. (2007). Teachers' gestures as a means of scaffolding students' understanding: evidence from an early algebra lesson. *Video Research in the Learning Sciences*, 349–365.

Bahrick, H. P., & Hall, L. K. (2005). The importance of retrieval failures to long-term retention: a metacognitive explanation of the spacing effect. *Journal of Memory and Language*, *52*, 566–577.

Ball, D. L. (1993). Halves, pieces, and twoths: constructing and using representational contexts in teaching fractions. *Rational Numbers: An Integration of Research*, 157–195.

Barbieri, C., & Booth, J. L. (2016). Support for struggling students in algebra: Contributions of incorrect worked examples. *Learning and Individual Differences, 48,* 36–44.

Begolli, K. N., & Richland, L. E. (2016). Teaching mathematics by comparison: analog visibility as a double-edged sword. *Journal of Educational Psychology*, *108*(2), 194–213.

Belenky, D. M., & Schalk, L. (2014). The effects of idealized and grounded materials on learning, transfer, and interest: an organizing framework for categorizing external knowledge representations. *Educational Psychology Review*, *26*(1), 27–50.

Belland, B. R., Walker, A. E., Olsen, M. W., & Leary, H. (2015). A pilot meta-analysis of computer-based scaffolding in STEM education. *Educational Technology & Society*, *18*(1), 183–197.

Booth, J. L., & Koedinger, K. R. (2012). Are diagrams always helpful tools? Developmental and individual differences in the effect of presentation format on student problem solving. *British Journal of Educational Psychology*, *82*(3), 492–511.

Booth, J. L., & Newton, K. J. (2012). Fractions: could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*, 247–253.

Booth, J. L., Lange, K. E., Koedinger, K. R., & Newton, K. J. (2013). Using example problems to improve student learning in algebra: differentiating between correct and incorrect examples. *Learning and Instruction*, *25*, 24–34.

Booth, J. L., Cooper, L. A., Donovan, M. S., Huyghe, A., Koedinger, K. R., & Paré-Blagoev, E. J. (2015a). Design-based research within the constraints of practice: AlgebraByExample. *Journal of Education for Students Placed at Risk*, *20*(1–2), 79–100.

Booth, J. L., Oyer, M. H., Paré-Blagoev, E. J., Elliot, A., Barbieri, C., Augustine, A. A., & Koedinger, K. R. (2015b). Learning algebra by example in real-world classrooms. *Journal of Research on Educational Effectiveness.*, *8*(4), 530–551.

Bottge, B. A. (1999). Effects of contextualized math instruction on problem solving of average and below-average achieving students. *Journal of Special Education*, *33*, 81–92.

Braithwaite, D. W., & Goldstone, R. L. (2013). Integrating formal and grounded representations in combinatorics learning. *Journal of Educational Psychology*, *105*(3), 666.

Braithwaite, D. W., & Goldstone, R. L. (2015). Effects of variation and prior knowledge on abstract concept learning. *Cognition and Instruction*, *33*(3), 226–256.

Budé, L., Imbos, T., van de Wiel, M. W., & Berger, M. P. (2011). The effect of distributed practice on students' conceptual understanding of statistics. *Higher Education*, *62*, 69–79.

Calder, N. (2015). Student wonderings: scaffolding student understanding within student-centered inquiry learning. *ZDM Mathematics Education*, *47*, 1121–1131.

Carpenter, S. K., Pashler, H., & Cepeda, N. J. (2009). Using tests to enhance 8th grade students' retention of U.S. history facts. *Applied Cognitive Psychology*, *23*, 760–771.

Carroll, W. M. (1994). Using worked examples as an instructional support in the algebra classroom. *Journal of Educational Psychology*, *86*(3), 360–367.

Carter, J. (1984). Instructional learner feedback: a literature review with implications for software development. *Computing Teacher*, *12*(2), 53–55.

Catrambone, R. (1996). Generalizing solution procedures learned from examples. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *22*(4), 1020–1031.

Catrambone, R. (1998). The subgoal learning model: creating better examples so that students can solve novel problems. *Journal of Experimental Psychology: General*, *127*(4), 355–376.

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. Availble from: http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Cepeda, N. J., Pashler, H., Vul, E., Wixted, J. T., & Rohrer, D. (2006). Distributed practice in verbal recall tasks: a review and quantitative synthesis. *Psychological Bulletin*, *132*, 354–380.

Challis, B. H. (1993). Spacing effects on cued-memory tests depend on level of processing. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *19*, 389–396.

Cherepinsky, V. (2011). Self-reflective grading: getting students to learn from their mistakes. *PRIMUS*, *21*(3), 294–301.

Cho, S., Holyoak, K. J., & Cannon, T. D. (2007). Analogical reasoning in working memory: resources shared among relational integration, interference resolution, and maintenance. *Memory & Cognition*, *35*(6), 1445–1455.

Cohen, V. B. (1985). A reexamination of feedback in computer-based instruction: implications for instructional design. *Educational Technology*, *25*(1), 33–37.

Cooper, G., & Sweller, J. (1987). Effects of schema acquisition and rule automation on mathematical problem-solving transfer. *Journal of Educational Psychology*, *79*(4), 347–362.

Cummins, D. D. (1992). Role of analogical reasoning in the induction of problem categories. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *18*(5), 1103.

Davenport, J. L., Klahr, D., & Koedinger, K. R. (2007). *The influence of diagrams on chemistry learning.* Paper presented at the Biennial Conference of the European Association for Research on Learning and Instruction, Hungary, Budapest.

Davenport, J. L., Lepori, K. L., Hauk, S., Viviani, K., & Schneider, S. A. (2012). Integration of Cognitive Design Principles with the Connected Mathematics Project Curriculum. Paper presented at the 2012 Annual Meeting of the American Educational Research Association in Vancouver, B.C.

De Bock, D., Deprez, J., Van Dooren, W., Roelens, M., & Verschaffel, L. (2011). Abstract or concrete examples in learning mathematics? A replication and elaboration of Kaminski, Sloutsky, and Heckler's study. *Journal for Research in Mathematics Education*, *42*(2), 109–126.

Delaney, P. F., Verkoeijen, P. P., & Spirgel, A. (2010). Spacing and testing effects: a deeply critical, lengthy, and at times discursive review of the literature. *Psychology of Learning and Motivation*, *53*, 63–147.

Dempster, F. N. (1988). The spacing effect: a case study in the failure to apply the results of psychological research. *American Psychologist*, *43*(8), 627.

Department of Education. (1997). *Mathematics equals opportunity*. White Paper prepared for U.S. Secretary of Education Richard W. Riley; Washington, DC.

D'Mello, S., & Graesser, A. (2014). Confusion and its dynamics during device comprehension with breakdown scenarios. *Acta Psychologica*, *151*, 106–116.

Dunlosky, J., Rawson, K. A., Marsh, E. J., Nathan, M. J., & Willingham, D. T. (2013). Improving students' learning with effective learning techniques promising directions from cognitive and educational psychology. *Psychological Science in the Public Interest*, *14*(1), 4–58.

Durkin, K., & Rittle-Johnson, B. (2012). The effectiveness of using incorrect examples to support learning about decimal magnitude. *Learning and Instruction*, *22*, 206–214.

Dynarski, M., Agodini, R., Heaviside, S., Novak, T., Carey, N., Campuzano, L., Means, B., …, & Sussex, W. (2007). *Effectiveness of reading and mathematics software products: findings from the first student cohort.* Report provided to Congress by the National Center for Education Evaluation.

Ellis, S., Klahr, D., & Siegler, R. S. (1993). Effects of feedback and collaboration on changes in children's use of mathematical rules. In *A paper presented in Society for Research in Child Development.* New Orleans.

Festinger, L. (1957). *A theory of cognitive dissonance*. Standford, CA: Stanford University Press.

Fleischman, H. L., Hopstock, P. J., Pelczar, M. P., & Shelley, B. E. (2010). Highlights from PISA 2009: Performance of US 15-Year-Old Students in Reading, Mathematics, and Science Literacy in an International Context. NCES 2011-004. *National Center for Education Statistics*.

Fyfe, E. R., Rittle-Johnson, B., & DeCaro, M. S. (2012). The effects of feedback during exploratory mathematics problem solving: prior knowledge matters. *Journal of Educational Psychology*, *104*(4), 1094–1108.

Fyfe, E. R., DeCaro, M. S., & Rittle-Johnson, B. (2014). An alternative time for telling: when conceptual instruction prior to problem solving improves mathematical knowledge. *British Journal of Educational Psychology*, 1–18.

Fyfe, E. R., DeCaro, M. S., & Rittle-Johnson, B. (2015a). When feedback is cognitively-demanding: the importance of working memory capacity. *Instructional Science*, *43*(1), 73–91.

Fyfe, E. R., McNeil, N. M., & Borjas, S. (2015b). Benefits of "concreteness fading" for children's mathematics understanding. *Learning and Instruction*, *35*, 104–120.

Gay, L. R. (1973). Temporal position of reviews and its effect on the retention of mathematical rules. *Journal of Educational Psychology*, *64*(2), 171–182.

Gentner, D. (1983). Structure-mapping: a theoretical framework for analogy. *Cognitive Science*, *7*(2), 155–170.

Gentner, D., & Rattermann, M. J. (1991). Language and the career of similarity. In *Perspectives on thought and language: Interrelations in development* (pp. 225–277).

Gick, M. L., & Holyoak, K. J. (1980). Analogical problem solving. *Cognitive Psychology*, *12*(3), 306–355.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *The Journal of the Learning Sciences*, *14*(1), 69–110.

Gorsky, P., & Finegold, M. (1994). The role of anomaly and of cognitive dissonance in restructuring students' concepts of force. *Instructional Science*, *22*, 75–90.

Graesser, A. C. (2009). Inaugural editorial for journal of educational psychology. *Journal of Educational Psychology*, *101*(2), 259–261.

Große, C. S., & Renkl, A. (2007). Finding and fixing errors in worked examples: can this foster learning outcomes? *Learning and Instruction*, *17*, 612–634.

Grossnickle, E. M., Dumas, D., Alexander, P. A., & Baggetta, P. (2016). Individual differences in the process of relational reasoning. *Learning and Instruction*, *42*, 141–159.

Guo, J. P., & Pang, M. F. (2011). Learning a mathematical concept from comparing examples: the importance of variation and prior knowledge. *European Journal of Psychology of Education*, *26*(4), 495–525.

Harks, B., Rakoczy, K., Hattie, J., Besser, M., & Klieme, E. (2014). The effects of feedback on achievement, interest and self-evaluation: the role of feedback's perceived usefulness. *Educational Psychology*, *34*(3), 269–290.

Hattikudur, S., & Alibali, M. W. (2010). Learning about the equal sign: does comparing with inequality symbols help? *Journal of Experimental Child Psychology*, *107*(1), 15–30.

Hausmann, R. G., & van Lehn, K. (2007). Explaining self-explaining: a contrast between content and generation. *Frontiers in Artificial Intelligence and Applications*, *158*, 417.

Heemsoth, T., & Heinze, A. (2014). The impact of incorrect examples on learning fractions: a field experiment with 6th grade students. *Instructional Science*, *42*(4), 639–657.

Hoven, J., & Garelick, B. (2007). Singapore Math: using the bar model approach, Singapore textbooks enable students to solve difficult math problems—and learn how to think symbolically. *Educational Leadership*, *65*, 38-21.

Hsiao, J., Hung, C., Lan, Y., & Jeng, Y. (2013). Integrating worked examples into problem posing in a web-based learning environment. *Turkish Online Journal of Educational Technology*, *12*(2), 166–176.

Janiszewski, C., Noel, H., & Sawyer, A. G. (2003). A meta-analysis of the spacing effect in verbal learning: implications for research on advertising repetition and consumer memory. *Journal of Consumer Research*, *30*, 138–149.

Kalyuga, S., Chandler, P., Tuovinen, J., & Sweller, J. (2001). When problem solving is superior to studying worked examples. *Journal of Educational Psychology*, *93*(3), 579–588.

Kaminski, J. A., & Sloutsky, V. M. (2013). Extraneous perceptual information interferes with children's acquisition of mathematical knowledge. *Journal of Educational Psychology*, *105*(2), 351.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2005). Relevant concreteness and its effects on learning and transfer. In *Proceedings of the XXVII annual conference of the Cognitive Science Society* (pp. 1090–1095).

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science*, 454–455.

Kay, R., & Edwards, J. (2012). Examining the use of worked example video podcasts in middle school mathematics classrooms: a formative analysis. *Canadian Journal of Learning and Technology*, *38*(3), .

Kehrer, P., Kelly, K., & Heffernan, N. (2013). Does immediate feedback while doing homework improve learning. In Youngblood Boonthum-Denecke (Ed.), *Proceedings of the Twenty-Sixth International Florida Artificial Intelligence Research Society Conference* (pp. 542–545). St. Pete Beach, Florida: AAAI Press.

Kim, M. C., & Hannafin, M. J. (2011). Scaffolding problem solving in technology-enhanced learning environments (TELEs): Bridging research and theory with practice. *Computers & Education*, *56*, 403–417.

Kim, D., Schneider, C., Engec, N., & Siskind, T. (2006). *South Carolina end-of-course examination program 2003–04 Algebra 1: Mathematics for the Technologies 2 state report*. Columbia, South Carolina: South Carolina Department of Education.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade-offs between grounded and abstract representations: evidence from algebra problem solving. *Cognitive Science*, *32*, 366–397.

Koedinger, K. R., Booth, J. L., & Klahr, D. (2013). Instructional complexity and the science to constrain it. *Science*, *342*, 935–937.

Krause, U. M., Stark, R., & Mandl, H. (2009). The effects of cooperative learning and feedback on e-learning in statistics. *Learning and Instruction*, *19*, 158–170.

Kulhavy, R. W. (1977). Feedback in written instruction. *Review of Educational Research*, *47*(1), 211–232.

Kupers, E., van Dijk, M., & van Geert, P. (2015). Within-teacher differences in one-to-one teacher—student interactions in instrumental music lessons. *Learning and Individual Differences*, *37*, 283–289.

Lannin, J., Townsend, B., & Barker, D. (2006). The reflective cycle of student error analysis. *For the Learning of Mathematics*, *26*(3), 33–38.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2006). *Connected Mathematics 2*. Pearson.

LeBlanc, K., & Simon, D. (2008). Mixed practice enhances retention and JOL accuracy for mathematical skills. Paper presented at the 49th Annual Meeting of the Psychonomic Society, Chicago, IL.

Lee, C., & Chen, M. (2015). Effects of worked examples using manipulatives on fifth graders' learning performance and attitude toward mathematics. *Journal of Educational Technology & Society*, *18*(1), 264–275.

Lehman, S., Schraw, G., McCrudden, M. T., & Hartley, K. (2007). Processing and recall of seductive details in scientific text. *Contemporary Educational Psychology*, *32*(4), 569–587.

Li, N., Cohen, W. W., & Koedinger, K. R. (2012). Problem order implications for learning transfer. In *Intelligent Tutoring Systems* (pp. 185–194). Springer Berlin Heidelberg.

Luwel, K., Foustana, A., Papadatos, Y., & Verschaffel, L. (2011). The role of intelligence and feedback in children's strategy competence. *Journal of Experimental Child Psychology*, *108*, 61–76.

Mayer, R., Sims, V. K., & Tajika, H. (1995). A comparison of how textbooks teach mathematical problem solving in Japan and the United States. *American Educational Research Journal*, *32*, 443–460.

Mayfield, K. H., & Chase, P. N. (2002). The effects of cumulative practice on mathematics problem solving. *Journal of Applied Behavior Analysis*, *35*(2), 105–123.

McLaren, B. M., Adams, D. M., & Mayer, R. E. (2015). Delayed learning effects with erroneous examples: a study of learning decimals with a web-based tutor. *International Journal of Artificial Intelligence in Education*, *25*(4), 520–542.

McNeil, N. M., & Fyfe, E. R. (2012). "Concreteness fading" promotes transfer of mathematical knowledge. *Learning and Instruction*, *22*(6), 440–448.

McNeil, N. M., & Uttal, D. H. (2009). Rethinking the use of concrete materials in learning: perspectives from development and education. *Child Development Perspectives*, *3*(3), 137–139.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. S., Hattikudur, S., et al. (2006). Middle school students' understanding of the equal sign: the books they read can't help. *Cognition and Instruction*, *24*, 367–385.

Mevarech, Z. R., & Kramarski, B. (2003). The effects of metacognitive training versus worked-out examples on students' mathematical reasoning. *British Journal of Educational Psychology*, *73*(4), 449–471.

Moreno, R., & Flowerday, T. (2006). Students' choice of animated pedagogical agents in science learning: a test of the similarity-attraction hypothesis on gender and ethnicity. *Contemporary Educational Psychology*, *31*, 186–207.

Mory, E. H. (2004). Feedback research revisited. In D. Jonassen (Ed.), *Handbook of research on educational communications and technology: a project for the Association for Educational Communications and Technology* (2nd., pp. 745–783). Mahwah, NJ: Lawrence Erlbaum Associates.

Mullis, I. V., Martin, M. O., Foy, P., & Arora, A. (2012). TIMSS 2011 international results in mathematics. International Association for the Evaluation of Educational Achievement, Amsterdam, The Netherlands.

NCLB. (2002). No Child Left Behind (NCLB) Act of 2001 (pp. 107–110). US Department of Education. Washington, DC.

National Mathematics Advisory Panel. (2008). *Foundations for success: The Final Report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education, 2008.

Novick, L. R., & Holyoak, K. J. (1991). Mathematical problem solving by analogy. *Journal of Experimental Psychology. Learning, Memory, and Cognition*, *17*(3), 398–415.

Ohlsson, S. (1996). Learning from performance errors. *Psychological Review*, *103*(2), 241.

Ostrow, K. S. & Heffernan, N. T. (2014). Testing the multimedia principle in the real world: a comparison of video vs. text feedback in authentic middle school math assignments. In J., Stamper, et al. (Eds.) Proceedings of the 7th International Conference on EDM. pp. 296–299.

Ostrow, K., Heffernan, N., Heffernan, C., & Peterson, Z. (2015). Blocking vs. interleaving: examining single-session effects within middle school math homework. In *Artificial Intelligence in Education* (pp. 338–347). Springer International Publishing.

Overoye, A. L., & Storm, B. C. (2015). Harnessing the power of uncertainty to enhance learning. *Translational Issues in Psychological Science*, *1*(2), 140–148.

Pashler, H., Bain, P., Bottge, B., Graesser, A., Koedinger, K., McDaniel, M., & Metcalfe, J. (2007). Organizing instruction and study to improve student learning: IES practice guide *(NCER 2007-2004)*. Washington, DC: National Center for Education Research.

Pennsylvania Department of Education (2011). 2011 Keystone Exams State Summary Report—Algebra I.

Petersen, L. A., & McNeil, N. M. (2013). Effects of perceptually rich manipulatives on preschoolers' counting performance: established knowledge counts. *Child Development*, *84*(3), 1020–1033.

Piaget, J. (1980). *Experiments in contradictions*. Chicago: University of Chicago Press.

Pratt, M. W., & Savoy-Levine, K. M. (1998). Contingent tutoring of long-divisions skills in fourth and fifth graders: experimental tests of some hypotheses about scaffolding. *Journal of Applied Developmental Psychology*, *19*, 287–304.

Rau, M. A., Aleven, V., & Rummel, N. (2013). Interleaved practice in multi-dimensional learning tasks: which dimension should we interleave? *Learning and Instruction*, *23*, 98–114.

Razzaq, L., & Heffernan, N. T. (2006). Scaffolding vs. hints in the assistment System. In *Intelligent Tutoring Systems* (pp. 635–644). Berlin, Heidelberg: Springer.

Rea, C. P., & Modigliani, V. (1985). The effect of expanded versus massed practice on the retention of multiplication facts and spelling lists. *Human Learning: Journal of Practical Research & Applications*, *4*(1), .

Reed, S. K., Corbett, A., Hoffman, B., Wagner, A., & MacLaren, B. (2013). Effect of worked examples and cognitive tutor training on constructing equations. *Instructional Science: An International Journal of The Learning Sciences*, *41*(1), 1–24.

Renkl, A., Stark, R., Gruber, H., & Mandl, H. (1998). Learning from worked-out examples: the effects of example variability and elicited self-explanations. *Contemporary Educational Psychology*, *23*(1), 90–108.

Retnowati, E., Ayres, P., & Sweller, J. (2010). Worked example effects in individual and group work settings. *Educational Psychology*, *30*(3), 349–367.

Richland, L. E., & Hansen, J. (2013). Reducing cognitive load in learning by analogy. *International Journal of Psychological Studies*, *5*(4), 69–80.

Richland, L. E., & McDonough, I. M. (2010). Learning by analogy: discriminating between potential analogs. *Contemporary Educational Psychology*, *35*, 28–43.

Richland, L. E., Holyoak, K. J., & Stigler, J. W. (2004). Analogy use in eighth-grade mathematics classrooms. *Cognition and Instruction*, *22*(1), 37–60.

Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogies in the mathematics classroom. *Science*, *316*(5828), 1128.

Rickard, T. C., Lau, J. S. H., & Pashler, H. (2008). Spacing and the transition from calculation to retrieval. *Psychonomic Bulletin & Review*, *15*(3), 656–661.

Rittle-Johnson, B., & Star, J. R. (2009). Compared with what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology*, *101*(3), 529–544.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2009). The importance of prior knowledge when comparing examples: influences on conceptual and procedural knowledge of equation solving. *Journal of Educational Psychology*, *101*(4), 836–852.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2012). Developing procedural flexibility: are novices prepared to learn from comparing procedures? *The British Journal of Educational Psychology*, *82*(Pt 3), 436–455.

Rohrer, D. (2009). The effects of spacing and mixing practice problems. *Journal for Research in Mathematics Education*, *40*, 4–17.

Rohrer, D. (2012). Interleaving helps students distinguish among similar concepts. *Educational Psychology Review*, *24*(3), 355–367.

Rohrer, D., & Taylor, K. (2006). The effects of overlearning and distributed practice on the retention of mathematics knowledge. *Applied Cognitive Psychology*, *20*, 1209–1224.

Rohrer, D., & Taylor, K. (2007). The shuffling of mathematics practice problems boosts learning. *Instructional Science*, *35*, 481–498.

Rohrer, D., Dedrick, R. F., & Burgess, K. (2014). The benefit of interleaved mathematics practice is not limited to superficially similar kinds of problems. *Psychonomic Bulletin & Review*, *21*, 1323–1330.

Rohrer, D., Dedrick, R. F., & Stershic, S. (2015). Interleaved practice improves mathematics learning. *Journal of Educational Psychology*, *107*(3), 900.

Ross, B. H. (1989). Distinguishing types of superficial similarities: different effects on the access and use of earlier problems. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *15*(3), 456.

Roy, M., & Chi, M. T. (2005). The self-explanation principle in multimedia learning. In R. E. Mayer (Ed.), *The Cambridge handbook of multimedia learning* (pp. 271–286). New York: Cambridge University Press.

Salden, R. J. C. M., Aleven, V. A. W. M. M., Renkl, A., & Schwonke, R. (2009). Worked examples and tutored problem solving: redundant or synergistic forms of support? *Topics in Cognitive Science*, *1*, 203–213.

Schutte, G. M., Duhon, G. J., Solomon, B. G., Poncy, B. C., Moore, K., & Story, B. (2015). A comparative analysis of massed vs. distributed practice on basic math fact fluency growth rates. *Journal of School Psychology*, *53*(2), 149–159.

Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and Instruction*, *16*(4), 475–5223.

Schwonke, R., Wittwer, J., Aleven, V., Salden, R.J.C.M., Krieg, C., & Renkl, A. (2007). Can tutored problem solving benefit from faded worked-out examples? Paper presented at *The European Cognitive Science Conference*, Delphi, Greece.

Schwonke, R., Renkl, A., Salden, R., & Aleven, V. (2011). Effects of different ratios of worked solution steps and problem solving opportunities on cognitive load and learning outcomes. *Computers in Human Behavior*, *27*, 58–62.

Shute, V. J. (2008). Focus on formative feedback. *Review of Educational Research*, *78*(1), 153–189.

Siegler, R. S. (2002). Microgenetic studies of self-explanations. In N. Granott, & J. Parziale (Eds.), *Microdevelopment: Transition processes in development and learning* (pp. 31–58). New York: Cambridge University.

Siegler, R. S., & Chen, Z. (2008). Differentiation and integration: guiding principles for analyzing cognitive change. *Developmental Science*, *11*(4), 433–453.

Siler, S. A., & Willows, K. J. (2014). Individual differences in the effect of relevant concreteness on learning and transfer of a mathematical concept. *Learning and Instruction*, *33*, 170–181.

Singh, R., Saleem, M., Pradhan, P., Heffernan, C., Heffernan, N., Razzaq, L. Dailey, M. O'Connor, C. & Mulchay, C. (2011). Feedback during Web-Based Homework: The Role of Hints. In Biswas, et al., (Eds.), *Proceedings of the Artificial Intelligence in Education Conference* (pp. 328–336).

Smit, J., van Eerde, H. A. A., & Bakker, A. (2013). A conceptualisation of whole-class scaffolding. *British Educational Research Journal*, *39*, 817–834.

Star, J. R., & Rittle-Johnson, B. (2009). Making algebra work: instructional strategies that deepen student understanding, within and between algebraic representations. *ERS Spectrum*, *27*(2), 11–18.

Star, J. R., Pollack, C., Durkin, K., Rittle-Johnson, B., Lynch, K., Newton, K., & Gogolen, C. (2015). Learning from comparison in algebra. *Contemporary Educational Psychology*, *40*, 41–54.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education*, *43*(4), 428–464.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: best ideas from the world's teachers for improving education in the classroom*. New York, NY: Free Press.

Sweller, J., & Cooper, G. (1985). The use of worked examples as a substitute for problem solving in learning algebra. *Cognition and Instruction*, *2*(1), 59–89.

Tarmizi, R. A., & Sweller, J. (1988). Guidance during mathematical problem solving. *Journal of Educational Psychology*, *80*(4), 424–436.

Taylor, K., & Rohrer, D. (2010). The effect of interleaving practice. *Applied Cognitive Psychology*, *24*, 837–848.

Thompson, C. A., & Opfer, J. E. (2010). How 15 hundred is like 15 cherries: effect of progressive alignment on representational changes in numerical cognition. *Child Development*, *81*(6), 1768–1786.

Uttal, D., Scudder, K., & DeLoache, J. (1997). Manipulatives as symbols: a new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology*, *19*, 37–54.

Uttal, D. H., Amaya, M., del Rosario Maita, M., Hand, L. L., Cohen, C. A., O'Doherty, K., & DeLoache, J. S. (2013). It works both ways: transfer difficulties between manipulatives and written subtraction solutions. *Child Development Research*, *2013*, Article ID 216367.

Vamvakoussi, X., & Vosniadou, S. (2012). Bridging the gap between the dense and the discrete: the number line and the "rubber line" bridging analogy. *Mathematical Thinking and Learning*, *14*, 265–284.

van de Pol, J., Volman, M., & Beishuizen, J. (2010). Scaffolding in teacher-student interaction: a decade of researcher. *Educational Psychology Review*, *22*, 271–296.

van Loon-Hillen, N., van Gog, T. V., & Brand-Gruwel, S. (2012). Effects of worked examples in a primary school mathematics curriculum. *Interactive Learning Environments*, *20*(1), 89–99.

Vorhölter, K., Kaiser, G., & Ferri, R. B. (2014). Modeling in mathematics classroom instruction: An innovative approach for transforming mathematics education. In *Transforming Mathematics Instruction* (pp. 21–36). Cham, Switzerland: Springer International Publishing.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Williams, S. E. (1997). Teachers' written comments and students' responses: A socially constructed interaction. *Paper presented at the Annual Meeting of the Conference on College Composition and Communication* (pp. 12–15). Phoenix, AZ.

Willingham, D. T. (2002). Ask the cognitive scientist allocating student study time "massed" versus "distributed" practice. *American Educator*, *26*(2), 37–39.

Wood, D., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, *17*(2), 89–100.

Zhu, X., & Simon, H. (1987). Learning mathematics from examples and by doing. *Cognition and Instruction*, *4*(3), 137–166.